

EXISTENCE AND CONVERGENCE THEOREMS FOR MULTIVALUED GENERALIZED HYBRID MAPPINGS IN CAT(κ)-SPACES

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ABSTRACT. In this study we give definition of some multivalued hybrid mappings which are general than multivalued nonexpansive mappings and some others. Also we give existence and convergence results in CAT(κ)-spaces

1. INTRODUCTION

Let K be a nonempty subset of a Hilbert space H and $T : K \rightarrow H$ be a mapping then, for all $x, y \in K$, if T satisfies

$$\|Tx - Ty\| \leq \|x - y\|,$$

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2,$$

and

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

then it called nonexpansive, nonspreading[4] and hybrid[17] respectively and none of this mappings included in other, In 2010 Aoyama et al.[6] defined λ -hybrid as a follows

$$(1 + \lambda)\|Tx - Ty\|^2 - \lambda\|x - Ty\|^2 \leq (1 - \lambda)\|x - y\|^2 + \lambda\|Tx - y\|^2$$

where $\lambda \in \mathbb{R}$. λ -hybrid mappings are general than nonexpansive mappings, nonspreading mappings and also hybrid mappings. In 2011 Aoyama and Kohsaka[7] introduced α -nonexpansive mappings in Banach spaces as follows,

$$\|Tx - Ty\|^2 \leq (1 - 2\alpha)\|x - y\|^2 + \alpha\|Tx - y\|^2 + \alpha\|x - Ty\|^2$$

where $\alpha < 1$ and also they showed that α -nonexpansive and λ -hybrid are equivalent in Hilbert spaces for $\lambda < 2$. Kocourek et al.[11] introduced more general mapping class than above mappings in Hilbert spaces, called (α, β) -generalized hybrid, as follows

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

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Also, in 2011, Lin et al.[9]defined a generalized hybrid mappings in $CAT(0)$, which is more general than nonexpansive, nonspreading and hybrid mappings in Banach spaces, as a follows

$$d^2(Tx, Ty) \leq a_1(x)d^2(x, y) + a_2(x)d^2(Tx, y) + a_3(x)d^2(x, Ty) \\ + k_1(x)d^2(Tx, x) + k_2(x)d^2(Ty, y)$$

where $a_1, a_2, a_3, k_1, k_2 : X \rightarrow [0, 1]$ with $a_1(x) + a_2(x) + a_3(x) < 1$, $2k_1(x) < 1 - a_2(x)$ and $2k_2(x) < 1 - a_3(x)$ for all $x, y \in X$. It is easy to see that (α, β) -generalized hybrid and generalized hybrid mappings are independent in metric spaces. In this paper we define two multivalued mapping class which general than (α, β) -generalized hybrid and generalized hybrid mappings, then establish existence and convergence theorems in $Cat(\kappa)$ spaces for $\kappa > 0$.

2. PRELIMINARIES

Let (X, d) be a metric space and K a nonempty subset of X . If there is $y \in K$ such that $d(x, y) = d(x, B) = \inf\{d(x, y); y \in B\}$ then K is called proximal subset of X . The family of nonempty compact convex subsets of X , the family of nonempty closed and convex subsets of X , the family of nonempty closed and bounded subsets of X and the family of nonempty proximal subsets will be denoted by $KC(X)$, $CC(X)$, $CB(X)$, $P(X)$, respectively. Let H Hausdorff Metric on $CB(X)$, defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\}$$

where $d(x, B) = \inf\{d(x, y); y \in B\}$.

A multivalued mappings $T : K \rightarrow 2^E$ is called nonexpansive if for all $x, y \in K$ and $p \in F(T)$

$$H(Tx, Ty) \leq d(x, y)$$

A point is called fixed point of T if $x \in Tx$ and the set of all fixed points of T is denoted by $F(T)$.

Many iterative method to approximate a fixed points of the mappings have been introduced for single-valued mappings in Banach spaces, well known one is defined by Picard as

$$x_{n+1} = Tx_n$$

After Picard, Mann and Ishikawa introduced new iteration procedures, respectively, as follows

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$$

and

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. In 2007, Agarwal *et al.* defined following iteration as

$$\begin{aligned} s_{n+1} &= (1 - \alpha_n)Ts_n + \alpha_n Tt_n \\ t_n &= (1 - \beta_n)s_n + \beta_n Ts_n \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. In 2008, S. Thianwan introduce two step iteration as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n Ty_n \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. M.A. Noor defined Noor iteration in 2001 and Phuengrattana and Suantai defined the SP iteration as follows

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n \\ y_n &= (1 - \beta_n)x_n + \beta_n Tz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n \end{aligned}$$

and

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n Ty_n \\ y_n &= (1 - \beta_n)z_n + \beta_n Tz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. Recently Renu Chugh, Vivek Kumar and Sanjay Kumar defined CR- iteration as follows

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n Ty_n \\ y_n &= (1 - \beta_n)Tx_n + \beta_n Tz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Also Gursoy and Karakaya [8] introduced Picard-S iteration as follows

$$\begin{aligned} x_{n+1} &= Ty_n \\ y_n &= (1 - \alpha_n)Tx_n \oplus \alpha_n Tz_n \\ z_n &= (1 - \beta_n)x_n \oplus \beta_n Tx_n \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

The multivalued version of Thianwan iteration and Picard-S iteration defined as follows

$$\begin{aligned} (2.1) \quad x_{n+1} &= P_K((1 - \alpha_n)y_n \oplus \alpha_n u_n) \\ y_n &= P_K((1 - \beta_n)x_n \oplus \beta_n v_n) \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $u_n \in Ty_n$, $v_n \in Tx_n$ and

$$(2.2) \quad \begin{aligned} x_{n+1} &= P_K(u_n) \\ y_n &= P_K((1 - \alpha_n)w_n \oplus \alpha_nv_n) \\ z_n &= P_K((1 - \beta_n)x_n \oplus \beta_nw_n) \end{aligned}$$

where $\{\alpha_n\}$ is a sequences in $[0, 1]$, $u_n \in Ty_n$, $v_n \in Tz_n$, $w_n \in Tx_n$.

Before the results we give some definitions and lemmas about $CAT(\kappa)$ with $\kappa > 0$ and Δ -convergences.

Let (X, d) bounded metric space, $x, y \in X$ and $C \subseteq X$ nonempty subset. A geodesic path (or shortly a geodesic) joining x and y is a map $c : [0, t] \subseteq \mathbb{R} \rightarrow X$ such that $c(0) = x$, $c(t) = y$ and $d(c(r), c(s)) = |r - s|$ for all $r, s \in [0, t]$. In particular c is an isometry and $d(c(0), c(t)) = t$. The image of c , $c([0, t])$ is called geodesic segment from x to y and it is unique (it not necessarily be unique) then it is denoted by $[x, y]$. $z \in [x, y]$ if and only if for an $t \in [0, 1]$ such that $d(z, x) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y)$. The point z is denoted by $z = (1 - t)x \oplus ty$. For fixed $r > 0$, the space (X, d) is called r -geodesic space if any two point $x, y \in X$ with $d(x, y) < r$ there is a geodesic joining x to y . if for every $x, y \in X$ there is a geodesic path then (X, d) called geodesic space and uniquely geodesic space if that geodesic path is unique for any pair x, y . A subset $C \subseteq X$ is called convex if it contains all geodesic segment joining any pair of points in it.

Definition 2.1. [5] Given a real number κ then

- i) if $\kappa = 0$ then M_κ^n is Euclidean space E^n
- ii) if $\kappa > 0$ then M_κ^n is obtained from the sphere S^n by multiplying distance function by $\frac{1}{\sqrt{\kappa}}$
- iii) if $\kappa < 0$ then M_κ^n is obtained from hyperbolic space H^n by multiplying distance function by $\frac{1}{\sqrt{-\kappa}}$

In geodesic metric space (X, d) , A geodesic triangle $\Delta(x, y, z)$ consist of three point x, y, z as vertices and three geodesic segments of any pair of these points, that is, $q \in \Delta(x, y, z)$ means that $q \in [x, y] \cup [x, z] \cup [y, z]$. The triangle $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ in M_κ^2 is called comparison triangle for the triangle $\Delta(x, y, z)$ such that $d(x, y) = d(\overline{x}, \overline{y})$, $d(x, z) = d(\overline{x}, \overline{z})$ and $d(y, z) = d(\overline{y}, \overline{z})$ and such a comparison triangle always exist provided that the perimeter $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ ($D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$ and ∞ otherwise) in M_κ^2 κ 2.14 in [5]. A point $\overline{z} \in [\overline{x}, \overline{y}]$ called comparison point for $z \in [x, y]$ if $d(x, z) = d(\overline{x}, \overline{z})$. A geodesic triangle $\Delta(x, y, z)$ in X with perimeter less than $2D_\kappa$ (and given a comparison triangle $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ for $\Delta(x, y, z)$ in M_κ^2) satisfies $CAT(\kappa)$ inequality if $d(p, q) \leq d(\overline{p}, \overline{q})$ for all $p, q \in \Delta(x, y, z)$ where $\overline{p}, \overline{q} \in \overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ are the comparison points of p, q respectively. The D_κ -geodesic metric space (X, d) is called $CAT(\kappa)$ space if every geodesic triangle in X with perimeter less than $2D_\kappa$ satisfies the $CAT(\kappa)$ inequality.

Bruhat and Tits [5] shows that If (X, d) is a $CAT(0)$ space,

$$d^2(x, \frac{1}{2}y \oplus \frac{1}{2}z) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z)$$

satisfied for every $x, y, z \in X$, its called *CN inequality* and its generalized by Dhompsonsa and Panyanak [5] as follows

$$d^2(x, (1 - \lambda)y \oplus \lambda z) \leq (1 - \lambda)d^2(x, y) + \lambda d^2(x, z) - \lambda(1 - \lambda)d^2(y, z)$$

for every $x, y, z \in X, \lambda \in [0, 1]$ (*CN* inequality*).

In fact, for a geodesic metric space (X, d) following three statements are equivalent;

- i) (X, d) is a *CAT(0)*
- ii) (X, d) satisfied *CN inequality*
- iii) (X, d) satisfied *CN* inequality*

Let (X, d) be geodesic space and $R \in (0, 2]$. if for every $x, y, z \in X$

$$d^2(x, (1 - \lambda)y \oplus \lambda z) \leq (1 - \lambda)d^2(x, y) + \lambda d^2(x, z) - \frac{R}{2}\lambda(1 - \lambda)d^2(y, z)$$

is satisfied then (X, d) called *R-convex* [10] Hence, (X, d) is a *CAT(0)* space if and only if it is a *2-convex space*

Proposition 2.2. [12] *The modulus of convexity for CAT(κ) space X (of dimension ≥ 2) and number $r < \frac{\pi}{2\sqrt{\kappa}}$ and let m denote the midpoint of the segment $[x, y]$ joining x and y define by the modulus δ_r by sitting*

$$\delta(r, \epsilon) = \inf\{1 - \frac{1}{r}d(a, m)\}$$

where the infimum is taken over all points $a, x, y \in X$ satisfying $d(a, x) \leq r$, $d(a, y) \leq r$ and $\epsilon \leq d(x, y) < \frac{\pi}{2\sqrt{\kappa}}$

Lemma 2.3. [15] *Let X be a complete CAT(κ)space with modulus of convexity $\delta(r, \epsilon)$ and let $x \in E$. Suppose that $\delta(r, \epsilon)$ increases with r (for a fixed ϵ) and suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$, $\{x_n\}$ and $\{y_n\}$ are the sequences in X such that $\limsup_{n \rightarrow \infty} d(x_n, x) \leq r$, $\limsup_{n \rightarrow \infty} d(y_n, x) \leq r$ and $\lim_{n \rightarrow \infty} d((1 - t_n)x_n \oplus t_n y_n, x) = r$ for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.*

Lemma 2.4. [13] *Let κ be an arbitrary positive real number and (X, d) be a CAT(κ) space with $\text{diam}(X) < \frac{\pi - \epsilon}{2\sqrt{\kappa}}$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then (X, d) is a R -convex space for $R = (\pi - 2\epsilon) \tan(\epsilon)$.*

Proposition 2.5. [5] *M_κ^n is a geodesic metric space. If $\kappa \leq 0$, then M_κ^n is uniquely geodesic and all balls in M_κ^n are convex. If $\kappa > 0$, then there is a unique geodesic segment joining x, y in M_κ^n if and only if $d(x, y) < \frac{\pi}{\sqrt{\kappa}}$. If $\kappa > 0$, closed balls in M_κ^n of radius smaller than $\frac{\pi}{2\sqrt{\kappa}}$ are convex.*

Proposition 2.6. [5] *Let X be $CAT(\kappa)$ space. Then any ball of radius smaller than $\frac{\pi}{2\sqrt{\kappa}}$ are convex.*

Proposition 2.7. *Exercise 2.3(1) in [5] Let $\kappa > 0$ and (X, d) be a $CAT(\kappa)$ space with $\text{diam}(X) < \frac{D_\kappa}{2} = \frac{\pi}{2\sqrt{\kappa}}$. Then, for any $x, y, z \in X$ and $t \in [0, 1]$, we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z).$$

Proposition 2.8. *\mathbb{R} -trees are particular class of $CAT(\kappa)$ spaces for any real number κ (see p.167 in [5]) and family of closed convex subsets of a $CAT(\kappa)$ spaces has uniform normal structure in usual metric sense.*

Definition 2.9. [5] An \mathbb{R} -tree is a metric space X such that

- i) it is a uniquely geodesic metric space,
- ii) if $x, y, z \in X$ are such that $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

Let $\{x_n\}$ be a bounded sequence in a $CAT(\kappa)$ space X and $x \in X$. Then, with setting

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n)$$

the asymptotic radius of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}); x \in X.\},$$

the asymptotic radius of $\{x_n\}$ with respect to $K \subseteq X$ is defined by

$$r_K(\{x_n\}) = \inf\{r(x, \{x_n\}); x \in K.\}$$

and the asymptotic center of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

and let $\omega_w(x_n) := \cup A(\{x_n\})$ where union is taken on all subsequences of $\{x_n\}$.

Definition 2.10. [3] A sequence $\{x_n\} \subset X$ is said to be Δ -convergent to $x \in X$ if x is the unique asymptotic center of all subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and read as x is Δ -limit of $\{x_n\}$.

Proposition 2.11. [3] *Let X be a complete $CAT(\kappa)$ space, $K \subseteq X$ nonempty, closed and convex, $\{x_n\}$ is a sequence in X . If $r_C(\{x_n\}) < \frac{\pi}{2\sqrt{\kappa}}$ then $A_C(\{x_n\})$ consist exactly one point.*

Lemma 2.12. [2]

- i) *Every bounded sequence in X has a Δ -convergent subsequence*
- ii) *If K is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in K , then the asymptotic center of $\{x_n\}$ is in K*
- iii) *If K is a closed convex subset of X and if $f : K \rightarrow X$ is a non-expansive mapping, then the conditions, $\{x_n\}$ Δ -converges to x and $\lim_{n \rightarrow \infty} d(x_n, f(x_n)) = 0$, imply $f(x) = x$ and $x \in K$.*

Lemma 2.13. [2] *If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = u$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$*

Definition 2.14. A multivalued mapping $T : K \rightarrow CB(K)$ is said to satisfy Condition (I) if there is a nondecreasing function $f : [0, \infty] \rightarrow [0, \infty]$ $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F))$ for all $x \in K$ where $F = F(T)$.

Definition 2.15. The mapping $T : X \rightarrow CB(X)$ is called hemicompact if, for any sequence $\{x_n\} \subset X$ such that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in X$

Lemma 2.16. [3] *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X . Then*

- i) the metric projection $P_K(x)$ of x onto K is a singleton,
- ii) if $x \notin K$ and $y \in K$ with $u \neq P_K(x)$, then $\angle_{P_K(x)}(x, y) \geq \frac{\pi}{2}$,
- iii) for each $y \in K$, $d(P_K(x), P_K(y)) \leq d(x, y)$.

Definition 2.17. T is called (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I from X to $CB(X)$ if

$$a_1(x)d^2(u, v) + a_2(x)d^2(u, y) \leq b_1(x)d^2(x, v) + b_2(x)d^2(x, y)$$

satisfied for all $x, y \in X$, $u \in Tx$ and $v \in Ty$ where $a_1, a_2 : X \rightarrow \mathbb{R} \setminus (0, 1)$ and $b_1, b_2 : X \rightarrow [0, 1]$ with $a_1(x) + a_2(x) \geq 1$ and $b_1(x) + b_2(x) \leq 1$.

Definition 2.18. T is called generalized multivalued hybrid mapping type I from X to $CB(X)$ if

$$\begin{aligned} d^2(u, v) \leq & a_1(x)d^2(x, y) + a_2(x)d^2(u, y) + a_3(x)d^2(x, v) \\ & + k_1(x)d^2(u, x) + k_2(x)d^2(v, y) \end{aligned}$$

for all $x, y \in X$, there are $u \in Tx$ and $y \in Ty$ where $a_1, a_2, a_3, k_1, k_2 : X \rightarrow [0, 1]$ with $a_1(x) + a_2(x) + a_3(x) < 1$, $2k_1(x) < 1 - a_2(x)$ and $2k_2(x) < 1 - a_3(x)$ for all $x \in X$.

Definition 2.19. T is called (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type II from X to $CB(X)$ if for all $x, y \in X$

$$a_1(x)H^2(Tx, Ty) + a_2(x)d^2(Tx, y) \leq b_1(x)d^2(x, Ty) + b_2(x)d^2(x, y)$$

where $a_1, a_2 : X \rightarrow \mathbb{R}$ and $b_1, b_2 : X \rightarrow \mathbb{R}$ with $a_1(x) + a_2(x) \geq 1$ and $b_1(x) + b_2(x) \leq 1$.

Definition 2.20. T is called generalized multivalued hybrid mapping type II from X to $CB(X)$ if for all $x, y \in X$

$$\begin{aligned} H^2(Tx, Ty) \leq & a_1(x)d^2(x, y) + a_2(x)d^2(Tx, y) + a_3(x)d^2(x, Ty) \\ & + k_1(x)d^2(Tx, x) + k_2(x)d^2(Ty, y) \end{aligned}$$

where $a_1, a_2, a_3, k_1, k_2 : X \rightarrow [0, 1]$ with $a_1(x) + a_2(x) + a_3(x) < 1$, $2k_1(x) < 1 - a_2(x)$ and $2k_2(x) < 1 - a_3(x)$ for all $x \in X$.

3. EXISTENCE RESULTS

Proposition 3.1. *Let X be a complete $CAT(\kappa)$ space, K be a nonempty, closed and convex subset of X with $\text{rad}(K) < \frac{\pi}{2\sqrt{\kappa}}$ and T be (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I from K to $C(K)$ with $F(T) \neq \emptyset$ then $F(T)$ closed and $Tp = \{p\}$ for all $p \in F(T)$*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ and $x_n \rightarrow x \in X$. Then for any $u \in Tx$, we have

$$\begin{aligned} d^2(u, x_n) &\leq a_1(x)d^2(u, x_n) + a_2(x)d^2(u, x_n) \\ &\leq b_1(x)d^2(x, x_n) + b_2(x)d^2(x, x_n) \\ &\leq d^2(x, x_n) \end{aligned}$$

then taking limit on n we have

$$d(u, x) \leq 0$$

so $u = x \in Tx = \{x\}$ □

Proposition 3.2. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space and K be a nonempty closed convex subset of X with $\text{rad}(K) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$ and $T : K \rightarrow C(X)$ be a generalized multivalued hybrid mapping type I with $F(T) \neq \emptyset$ then $F(T)$ closed and $Tp = \{p\}$ for all $p \in F(T)$*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ and $x_n \rightarrow x \in X$. Then for any $u \in Tx$, we have

$$\begin{aligned} d^2(u, x_n) &\leq a_1(x)d^2(x, x_n) + a_2(x)d^2(u, x_n) + a_3(x)d^2(x, x_n) \\ &\quad + k_1(x)d^2(u, x) + k_2(x)d^2(x_n, x_n) \end{aligned}$$

implies that

$$d(u, x_n) \leq d^2(x, x_n) + \frac{k_1(x)}{1 - a_2(x)} d^2(u, x)$$

then taking limit on n we have

$$(1 - \frac{k_1(x)}{1 - a_2(x)})d(u, x) \leq 0$$

so $u = x \in Tx = \{x\}$ □

Proposition 3.3. *Let X be a complete $CAT(\kappa)$ space, K be a nonempty, closed and convex subset of X with $\text{rad}(K) < \frac{\pi}{2\sqrt{\kappa}}$ and T be (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type II from K to $C(K)$ with $F(T) \neq \emptyset$ and $a_1(p) \geq 1$ for all $p \in F(T)$ then $F(T)$ closed and $Tp = \{p\}$ for all $p \in F(T)$*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ and $x_n \rightarrow x \in X$. Then for any Tx , we have

$$\begin{aligned} d^2(Tx, x_n) &\leq a_1(x)d^2(Tx, x_n) + a_2(x)d^2(Tx, x_n) \\ &\leq a_1(x)H^2(Tx, Tx_n) + a_2(x)d^2(Tx, x_n) \\ &\leq b_1(x)d^2(x, Tx_n) + b_2(x)d^2(x, x_n) \\ &\leq d^2(x, x_n) \end{aligned}$$

then taking limit on n we have

$$d(Tx, x) \leq 0$$

so $u = x \in Tx = \{x\}$ □

Proposition 3.4. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space and K be a nonempty closed convex subset of X with $rad(K) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$ and $T : K \rightarrow KC(X)$ be a generalized multivalued hybrid mapping type II with $F(T) \neq \emptyset$ then $F(T)$ closed and $Tp = \{p\}$ for all $p \in F(T)$*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ and $x_n \rightarrow x \in X$. Then for any $u \in Tx$, we have we can find $y_n \in Tx_n$ such that $d(u, y_n) = d(u, Tx_n) \cdot d(x_n, y_n) \leq d \lim_{n \rightarrow \infty} d$

$$\begin{aligned} d^2(u, Tx_{n-1}) &\leq H^2(Tx, Tx_{n-1}) \\ &\leq a_1(x)d^2(x, x_{n-1}) + a_2(x)d^2(Tx, x_{n-1}) + a_3(x)d^2(x, Tx_{n-1}) \\ &\quad + k_1(x)d^2(Tx, x) + k_2(x)d^2(Tx_{n-1}, x_n) \\ &\leq a_1(x)d^2(x, x_{n-1}) + a_2(x)d^2(u, x_{n-1}) + a_3(x)d^2(x, x_n) \\ &\quad + k_1(x)d^2(u, x) \end{aligned}$$

implies that

$$d^2(u, x_n) \leq d^2(x, x_n) + \frac{k_1(x)}{1 - a_2(x)}d^2(u, x)$$

then taking limit on n we have

$$(1 - \frac{k_1(x)}{1 - a_2(x)})d(u, x) \leq 0$$

so $u = x \in Tx = \{x\}$ □

Theorem 3.5. *Let X be a complete $CAT(\kappa)$ space, K be a nonempty, closed and convex subset of X with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$ and T be (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I from K to $C(K)$ then $F(T) \neq \emptyset$.*

Proof. Let $x_0 \in K$ and $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$. Assume that $A_C\{x_n\} = \{z\}$ then $z \in K$ by Lemma 2.12. Since T is type I, for all $n \in \mathbb{N}$ and for any $u \in Tz$ such that

$$a_1(z)d^2(u, x_n) + a_2(z)d^2(u, x_{n-1}) \leq b_1(z)d^2(z, x_n) + b_2(z)d^2(z, x_{n-1})$$

and taking limit superior on both side which implies that

$$\limsup_{n \rightarrow \infty} d^2(u, x_n) \leq \limsup_{n \rightarrow \infty} d^2(z, x_n).$$

hencet $z = u \in Tz = \{u\}$. \square

Corollay 3.6. *Let X be a complete $CAT(\kappa)$ space, K be a nonempty closed convex subset of X with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$ and T be (a_1, a_2, b_1, b_2) - hybrid mapping from K to K then $F(T) \neq \emptyset$.*

Proof. Let take $F = \{T(x)\}$, then F is (a_1, a_2, b_1, b_2) -multivalued hybrid mapping from K to $C(K)$. Hence F has at least one fixed point and so T by Theorem 3.5 \square

Since for any $\kappa > \kappa'$, $CAT(\kappa')$ space is $CAT(\kappa)$ then following corollaries holds.

Corollay 3.7. *Let X be a complete $CAT(0)$ space, K be a nonempty closed convex subset of X and T be (a_1, a_2, b_1, b_2) - multivalued hybrid mapping type I from K to $C(K)$. then there is a $x_0 \in K$ such that the sequence $\{x_n\}$ defined by $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ is bounded if and only if $F(T) \neq \emptyset$.*

Corollay 3.8. *Let X be a complete $CAT(0)$ space, K be a nonempty closed convex subset of X and T be (a_1, a_2, b_1, b_2) -hybrid mapping from K to K . then there is a $x_0 \in K$ such that the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ is bounded if and only if $F(T) \neq \emptyset$.*

Theorem 3.9. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space and K be a nonempty closed convex subset of X with $rad(K) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$ and $T : K \rightarrow C(K)$ be a generalized multivalued hybrid mapping type I with $k_1(x) = k_2(x) = 0$ for all $x \in K$ then $F(T) \neq \emptyset$.*

Proof. Let $x_0 \in K$ and $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$. Assume that $A_C\{x_n\} = \{z\}$ then $z \in K$ by Lemma 2.12. Since T is type I, for all $n \in \mathbb{N}$ we can find a $u \in Tz$ such that multivalued hybrid mapping

$$d^2(x_n, u) \leq a_1(z)d^2(x_{n-1}, z) + a_2(z)d^2(x_{n-1}, u) + a_3(z)d^2(x_n, z)$$

and by taking limit superior on both side we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} d^2(x_n, u) &\leq (a_1(z) + a_3(z)) \limsup_{n \rightarrow \infty} d^2(x_n, z) \\ &\quad + a_2(z) \limsup_{n \rightarrow \infty} d^2(x_{n-1}, u) \end{aligned}$$

implies that $\limsup_{n \rightarrow \infty} d(x_n, u) \leq \limsup_{n \rightarrow \infty} d(x_n, z)$ which implies that $z = u \in Tz$. \square

Corollary 3.10. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space, K be a nonempty closed convex subset of X with $\text{rad}(K) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$ and $T : K \rightarrow K$ be a generalized hybrid mapping type I with $k_1(x) = k_2(x) = 0$ for all $x \in K$ then $F(T) \neq \emptyset$.*

Proof. Let take $F = \{T(x)\}$, then F is multivalued hybrid mapping from K to $C(K)$. Hence F has at least one fixed point and so T by Theorem 3.9. \square

Corollary 3.11. *Let X be a complete $CAT(0)$ space and K be a nonempty closed convex subset of X , and $T : K \rightarrow K(K)$ be a generalized multivalued hybrid mapping type I with $k_1(x) = k_2(x) = 0$ for all $x \in K$ then there is a $x_0 \in K$ such that the sequence $\{x_n\}$ defined by $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ is bounded if and only if $F(T) \neq \emptyset$*

Corollary 3.12. *Let X be a complete $CAT(0)$ space and K be a nonempty closed convex subset of X , and $T : K \rightarrow K$ be a generalized hybrid mapping with $k_1(x) = k_2(x) = 0$ for all $x \in K$. Then there is a $x_0 \in K$ such that the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ is bounded if and only if $F(T) \neq \emptyset$.*

Theorem 3.13. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow K(K)$ be a generalized multivalued hybrid mapping type II satisfying either*

- i) $a_2(x) = 0$ and $\frac{2k_2(x)}{1-a_3(x)} < \frac{R}{2}$ or
- ii) $a_3(x) = 0$ and $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$

for all $x \in K$ where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Moreover, If $k = \sup \frac{a_1(x)+k_1(x)}{1-k_2(x)} < 1$ then $F(T) \neq \emptyset$.

Proof. Let $x_0 \in K$ and $x_{n+1} \in Tx_n$ such that $d(x_{n+1}, x_n) = d(Tx_n, x_n)$ for all $n \in \mathbb{N}$. Assume that $a_2(x) = 0$. Then

$$\begin{aligned} d^2(x_{n+1}, x_n) &= d^2(Tx_n, x_n) \leq H^2(Tx_n, Tx_{n-1}) \\ &\leq a_1(x_n)d^2(x_n, x_{n-1}) + a_3(x)d^2(Tx_{n-1}, x_n) \\ &\quad + k_1(x_n)d^2(Tx_n, x_n) + k_2(x_n)d^2(Tx_{n-1}, x_{n-1}) \\ &\leq a_1(x_n)d^2(x_n, x_{n-1}) + k_1(x_n)d^2(Tx_n, x_n) \\ &\quad + k_2(x_n)d^2(Tx_{n-1}, x_{n-1}) \end{aligned}$$

implies that

$$d^2(x_{n+1}, x_n) \leq \frac{a_1(x_n) + k_1(x_n)}{1 - k_2(x_n)} d^2(x_n, x_{n-1})$$

hence we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \sqrt{\frac{a_1(x_n) + k_1(x_n)}{1 - k_2(x_n)}} d(x_n, x_{n-1}) \\ &\leq k d(x_n, x_{n-1}) \\ &\leq k^n d(x_1, x_0). \end{aligned}$$

Let $n < m$, then

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \\ &\leq \sum_{i=n}^{m-1} k^{i+1} d(x_1, x_0) \\ &\leq d(x_1, x_0) \sum_{i=n}^{m-1} k^{i+1}. \end{aligned}$$

Since $k < 1$ then the sequence (x_n) is Cauchy sequence and since space is complete than $x_n \rightarrow z \in X$.

$$\begin{aligned} d^2(x_n, Tz) &\leq H^2(Tx_{n-1}, Tz) \\ &\leq a_1(z) d^2(x_{n-1}, z) + a_3(z) d^2(x_{n-1}, Tz) \\ &\quad + k_1(z) d^2(Tx_{n-1}, x_{n-1}) + k_2(z) d^2(Tz, z) \end{aligned}$$

on the other hand

$$d^2(x_n, \frac{1}{2}Tz \oplus \frac{1}{2}z) \leq \frac{1}{2}d^2(x_n, Tz) + \frac{1}{2}d^2(x_n, z) - \frac{R}{8}d^2(z, Tz)$$

which implies that

$$d^2(z, Tz) \leq \frac{4}{R}d^2(x_n, Tz) + \frac{4}{R}d^2(x_n, z)$$

so we we have

$$\begin{aligned} d^2(x_n, Tz) &\leq a_1(z) d^2(x_{n-1}, z) + a_3(z) d^2(x_{n-1}, Tz) \\ &\quad + k_1(z) d^2(Tx_{n-1}, x_{n-1}) + k_2(z) d^2(Tz, z) \\ &\leq a_1(z) d^2(x_{n-1}, z) + a_3(z) d^2(x_{n-1}, Tz) + k_1(z) d^2(Tx_{n-1}, x_{n-1}) \\ &\quad + k_2(z) \left(\frac{4}{R} d^2(x_n, Tz) + \frac{4}{R} d^2(x_n, z) \right) \end{aligned}$$

which implies that

$$(1 - k_2(z) \frac{4}{R} - a_3(z)) \lim_{n \rightarrow \infty} d^2(x_n, Tz) \leq 0$$

so $\lim_{n \rightarrow \infty} d^2(x_n, Tz) = 0$. Hence $d^2(z, Tz) \leq d^2(x_n, z) + d^2(x_n, Tz) \rightarrow \infty$ implies that $z \in Tz$. \square

4. CONVERGENCE RESULTS

Theorem 4.1. *(Demiclosed principle for (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I) Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow C(X)$ be a (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I. Let $\{x_n\}$ be a sequence in K with $\Delta - \lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $z \in T(z)$.*

Proof. By Lemma 2.12, $z \in K$. We can find a sequence $\{y_n\}$ such that $d(x_n, y_n) = d(x_n, Tx_n)$, so we have $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. Because of T is (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I, for all $u \in Tz$ such that

$$a_1(z)d^2(u, y_n) + a_2(z)d^2(u, x_n) \leq b_1(z)d^2(z, y_n) + b_2(z)d^2(z, x_n)$$

Then by triangular inequality we have $d(x_n, u) \leq d(x_n, y_n) + d(y_n, u)$. So we have, $\limsup_{n \rightarrow \infty} d(x_n, u) \leq \limsup_{n \rightarrow \infty} d(y_n, u)$ and again since $d(y_n, u) \leq d(y_n, x_n) + d(x_n, u)$ we have $\limsup_{n \rightarrow \infty} d(y_n, u) \leq \limsup_{n \rightarrow \infty} d(x_n, u)$, combining these we have that $\limsup_{n \rightarrow \infty} d(x_n, u) = \limsup_{n \rightarrow \infty} d(y_n, u)$. So we have that

$$\begin{aligned} a_1(z)d^2(u, y_n) + a_2(z)d^2(u, x_n) &\leq b_1(z)d^2(z, y_n) + b_2(z)d^2(z, x_n) \\ &\leq b_1(z)[d(z, x_n) + d(x_n, y_n)]^2 + b_2(z)d^2(x_n, z) \end{aligned}$$

which implies that $\limsup_{n \rightarrow \infty} d(u, x_n) \leq \limsup_{n \rightarrow \infty} d(z, x_n)$. Then $z = u \in Tz$. Assume that $a_2(z) > 0$. \square

Corollary 4.2. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow X$ be a (a_1, a_2, b_1, b_2) -hybrid mapping. Let $\{x_n\}$ be a sequence in K with $\Delta - \lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $Tz = z$.*

Corollary 4.3. *Let X be a complete $CAT(0)$ space and K be a nonempty closed convex subset of X , and $T : K \rightarrow C(X)$ be a (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I. Let $\{x_n\}$ be a bounded sequence in K with $\Delta - \lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $z \in Tz$.*

Corollary 4.4. *Let X be a complete $CAT(0)$ space and K be a nonempty closed convex subset of X , and $T : K \rightarrow X$ be a (a_1, a_2, b_1, b_2) -hybrid mapping. Let $\{x_n\}$ be a bounded sequence in K with $\Delta - \lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $Tz = z$.*

Theorem 4.5. *(Demiclosed principle for generalized multivalued hybrid mapping type I) Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow C(X)$ be a generalized multivalued hybrid mapping type I with $\frac{2k_1(x)}{1 - a_2(x)} < \frac{R}{2}$ for all $x \in K$ where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Let $\{x_n\}$ be a sequence in K with $\Delta - \lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $z \in T(z)$.*

Proof. By Lemma 2.12, $z \in K$. We can find a sequence $\{y_n\}$ such that $d(x_n, y_n) = d(x_n, Tx_n)$, so we have $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. Since T is generalized multivalued hybrid mapping type I, for all $u \in Tz$ such that,

$$\begin{aligned} d^2(y_n, u) &\leq a_1(z)d^2(x_n, z) + a_2(z)d^2(u, x_n) + a_3(z)d^2(y_n, z) \\ &\quad + k_1(z)d^2(y_n, x_n) + k_2(x)d^2(u, z) \\ &\leq a_1(z)d^2(x_n, z) + a_2(z)[d(x_n, y_n) + d(y_n, u)]^2 \\ &\quad + a_3(z)[d(y_n, x_n) + d(x_n, z)]^2 \\ &\quad + k_1(z)d^2(y_n, x_n) + k_2(x)d^2(u, z) \end{aligned}$$

which implies that

$$\limsup_{n \rightarrow \infty} d^2(y_n, u) \leq \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{k_2(x)}{1 - a_2(x)} d^2(z, u)$$

and using this we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} d^2(x_n, u) &\leq \limsup_{n \rightarrow \infty} [d^2(x_n, y_n) + d^2(y_n, u)] \\ &\leq \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{k_1(x)}{1 - a_2(x)} d^2(z, u). \end{aligned}$$

By CN^x inequality we have

$$d^2(x_n, \frac{1}{2}z \oplus \frac{1}{2}u) \leq \frac{1}{2}d^2(x_n, z) + \frac{1}{2}d^2(x_n, u) - \frac{R}{8}d^2(z, u)$$

and combining all of these we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} d^2(x_n, \frac{1}{2}z \oplus \frac{1}{2}u) &\leq \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, u) \\ &\quad - R \frac{1}{8} d^2(z, u) \\ &\leq \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) \\ &\quad + \frac{k_1(x)}{2(1 - a_2(x))} d^2(z, u) - \frac{R}{8} d^2(z, u). \\ &= \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) \\ &\quad + (\frac{k_1(x)}{2(1 - a_2(x))} - \frac{R}{8}) d^2(z, u) \\ &= \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) \\ &\quad + (\frac{k_1(x)}{2(1 - a_2(x))} - \frac{R}{8}) d^2(z, u) \end{aligned}$$

which implies that

$$(\frac{R}{8} - \frac{k_1(x)}{2(1 - a_2(x))}) d^2(z, u) \leq \limsup_{n \rightarrow \infty} d^2(x_n, z) - \limsup_{n \rightarrow \infty} d^2(x_n, \frac{1}{2}z \oplus \frac{1}{2}u) \leq 0$$

and by assumptions we have $z = u \in Tz$ \square

Corollary 4.6. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow X$ be a generalized hybrid mapping with $\frac{2k_1(x)}{1-a_2(x)} < R$ for all $x \in K$ where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Let $\{x_n\}$ be a sequence in K with $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $T(z) = z$.*

Corollary 4.7. *Let X be a complete $CAT(0)$ space and K be a nonempty closed convex subset of X , and $T : K \rightarrow C(X)$ be a generalized multivalued hybrid mapping type I. Let $\{x_n\}$ be a sequence in K with $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $z \in T(z)$.*

Corollary 4.8. *Let X be a complete $CAT(0)$ space and K be a nonempty closed convex subset of X , and $T : K \rightarrow X$ be a generalized hybrid mapping. Let $\{x_n\}$ be a sequence in K with $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $Tz = z$.*

Theorem 4.9. *(Demiclosed principle for generalized multivalued hybrid mapping type II) Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow KC(X)$ be a generalized multivalued hybrid mapping with $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ for all $x \in K$ where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Let $\{x_n\}$ be a sequence in K with $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $z \in T(z)$.*

Proof. By Lemma 2, $z \in K$. We can find a sequence $\{y_n\}$ such that $y_n \in Tx_n$, $d(x_n, y_n) = d(x_n, Tx_n)$, so we have $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ and since Tz is compact we can find a sequence $\{z_n\}$ in Tz such that $d(y_n, z_n) = d(y_n, Tz)$. Then there is a convergent subsequence $\{z_{n_i}\}$ of $\{z_n\}$, say $\lim_{i \rightarrow \infty} z_{n_i} = u \in Tz$.

$$\begin{aligned} d(x_{n_i}, u) &\leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, z_{n_i}) + d(z_{n_i}, u) \\ &\leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, Tz) + d(z_{n_i}, u) \\ &\leq d(x_{n_i}, y_{n_i}) + H(Tx_{n_i}, Tz) + d(z_{n_i}, u) \\ &\leq d(x_{n_i}, y_{n_i}) + H(Tx_{n_i}, Tz) + d(z_{n_i}, u) \end{aligned}$$

implies that $\limsup_{n \rightarrow \infty} d(x_{n_i}, u) \leq \limsup_{n \rightarrow \infty} H(Tx_n, Tz)$. Because of T is generalized multivalued hybrid mapping,

$$\begin{aligned} H^2(Tx_n, Tz) &\leq a_1(z)d^2(x_n, z) + a_2(z)d^2(Tz, x_n) + a_3(z)d^2(Tx_n, z) \\ &\quad + k_1(z)d^2(Tx_n, x_n) + k_2(x)d^2(Tz, z) \\ &\leq a_1(z)d^2(x_n, z) + a_2(z)[d(x_n, Tx_n) + H(Tx_n, Tz)]^2 \\ &\quad + a_3(z)[d(Tx_n, x_n) + d(x_n, z)]^2 + k_1(z)d^2(Tx_n, x_n) + k_2(x)d^2(Tz, z) \end{aligned}$$

which implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} H^2(Tx_n, Tz) &\leq \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{k_1(x)}{1 - a_2(x)} d^2(z, Tz) \\ &\leq \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{k_1(x)}{1 - a_2(x)} d^2(z, u). \end{aligned}$$

By CN^x inequality we have

$$d^2(x_n, \frac{1}{2}z \oplus \frac{1}{2}u) \leq \frac{1}{2}d^2(x_n, z) + \frac{1}{2}d^2(x_n, u) - \frac{R}{8}d^2(z, u)$$

and combining all of these we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} d^2(x_n, \frac{1}{2}z \oplus \frac{1}{2}u) &\leq \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, u) - \frac{R}{8}d^2(z, u) \\ &\leq \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{1}{2} \limsup_{n \rightarrow \infty} H(Tx_n, Tz) - \frac{R}{8}d^2(z, u). \\ &\leq \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) + \frac{1}{2} \limsup_{n \rightarrow \infty} d^2(x_n, z) \\ &\quad + \frac{k_1(x)}{2(1 - a_2(x))} d^2(z, u) - \frac{R}{8}d^2(z, u). \\ &= \limsup_{n \rightarrow \infty} d^2(x_n, z) + (\frac{k_1(x)}{2(1 - a_2(x))} - \frac{R}{8})d^2(z, u) \end{aligned}$$

which implies that

$$(\frac{R}{8} - \frac{k_1(x)}{2(1 - a_2(x))})d^2(z, u) \leq \limsup_{n \rightarrow \infty} d^2(x_n, z) - \limsup_{n \rightarrow \infty} d^2(x_n, \frac{1}{2}z \oplus \frac{1}{2}u) \leq 0$$

and by assumptions we have $z = u \in Tz$. \square

Corollary 4.10. *Let X be a complete $CAT(0)$ space and K be a nonempty closed convex subset of X , and $T : K \rightarrow KC(X)$ be a generalized multivalued hybrid mapping type II. Let $\{x_n\}$ be a sequence in K with $\Delta - \lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $z \in T(z)$.*

Theorem 4.11. *(Demiclosed principle for (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type II) Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty convex compact subset of X , and $T : K \rightarrow KC(X)$ be a (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type II with $a_1(x) \geq 1$ for all $x \in K$. Let $\{x_n\}$ be a sequence in K with $\Delta - \lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $z \in T(z)$.*

Proof. By Lemma 2.12, $z \in K$. We can find a sequence $\{y_n\}$ such that $y_n \in Tx_n$, $d(x_n, y_n) = d(x_n, Tx_n)$, so we have $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ and since Tz is compact we can find a sequence $\{z_n\}$ in Tz such that $d(y_n, z_n) = d(y_n, Tz)$. Then there is a convergent subsequence $\{z_{n_i}\}$ of $\{z_n\}$, say $\lim_{i \rightarrow \infty} z_{n_i} = u \in Tz$.

$$\begin{aligned}
d(x_{n_i}, u) &\leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, z_{n_i}) + d(z_{n_i}, u) \\
&\leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, Tz) + d(z_{n_i}, u) \\
&\leq d(x_{n_i}, y_{n_i}) + H(Tx_{n_i}, Tz) + d(z_{n_i}, u) \\
&\leq d(x_{n_i}, y_{n_i}) + H(Tx_{n_i}, Tz) + d(z_{n_i}, u)
\end{aligned}$$

implies that $\limsup_{n \rightarrow \infty} d(x_{n_i}, u) \leq \limsup_{n \rightarrow \infty} H(Tx_{n_i}, Tz)$. Because of T is (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type II,

$$a_1(z)H^2(Tz, Tx_{n_i}) \leq b_1(z)d^2(z, x_{n_i}) + b_2(z)d^2(z, x_{n_i}) - a_2(z)d^2(z, Tx_{n_i})$$

Then by triangular inequality we have $d(x_n, u) \leq d(x_n, y_n) + d(y_n, u)$. So we have, $\limsup_{n \rightarrow \infty} d(x_n, u) \leq \limsup_{n \rightarrow \infty} d(y_n, u)$ and again since $d(y_n, u) \leq d(y_n, x_n) + d(x_n, u)$ we have $\limsup_{n \rightarrow \infty} d(y_n, u) \leq \limsup_{n \rightarrow \infty} d(x_n, u)$, combining these we have that $\limsup_{n \rightarrow \infty} d(x_n, u) = \limsup_{n \rightarrow \infty} d(y_n, u)$. So we have that

$$\begin{aligned}
&\leq a_1(z)d^2(y_n, u) + a_2(z)d^2(x_n, u) \leq b_1(z)d^2(y_n, z) + b_2(z)d^2(x_n, z) \\
&\leq b_1(z)[d(y_n, x_n) + d(x_n, z)]^2 + b_2(z)d^2(x_n, z)
\end{aligned}$$

which implies that $\limsup_{n \rightarrow \infty} d(x_n, u) \leq \limsup_{n \rightarrow \infty} d(x_n, z)$ Then $z = u \in Tz$. Assume that $a_2(z) > 0$. \square

Lemma 4.12. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow C(X)$ be a generalized multivalued hybrid mapping type I with $\frac{2k_1(x)}{1-a_2(x)} < R$ for all $x \in K$ where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Let $\{x_n\}$ be a sequence in K with $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, p)\}$ converges for all $p \in F(T)$. Then $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ include exactly one point.*

Proof. Let take $u \in \omega_w(x_n)$ then there exist subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. Then By Lemma 2.12 there exist subsequence $\{v_n\}$ of $\{u_n\}$ with $\Delta - \lim_{n \rightarrow \infty} v_n = v \in K$. Then by Theorem 4.1 we have $v \in F(T)$ and by Lemma 2.13 we conclude that $u = v$, hence we get $\omega_w(x_n) \subseteq F(T)$. Let take subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. Because of $v \in \omega_w(x_n) \subseteq F(T)$, $\{d(x_n, u)\}$ converges, so by Lemma 2.13 we have $x = u$, this means that $\omega_w(x_n)$ include exactly one point. \square

Lemma 4.13. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow C(X)$ be (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I. Let $\{x_n\}$ be a sequence in K with $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, p)\}$ converges for all $p \in F(T)$. Then $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ include exactly one point.*

Proof. Let take $u \in \omega_w(x_n)$ then there exist subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. Then By Lemma 2.12 there exist subsequence $\{v_n\}$ of $\{u_n\}$ with $\Delta - \lim_{n \rightarrow \infty} v_n = v \in K$. Then by Theorem 4.5 we have $v \in F(T)$

and by Lemma 2.13 we conclude that $u = v$, hence we get $\omega_w(x_n) \subseteq F(T)$. Let take subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. Because of $v \in \omega_w(x_n) \subseteq F(T)$, $\{d(x_n, u)\}$ converges, so by Lemma 2.13 we have $x = u$, this means that $\omega_w(x_n)$ include exactly one point. \square

Lemma 4.14. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow KC(X)$ be a generalized multivalued hybrid mapping type II with $\frac{2k_1(x)}{1-a_2(x)} < R$ for all $x \in K$ where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Let $\{x_n\}$ be a sequence in K with $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, p)\}$ converges for all $p \in F(T)$. Then $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ include exactly one point.*

Proof. Let take $u \in \omega_w(x_n)$ then there exist subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. Then By Lemma 2.12 there exist subsequence $\{v_n\}$ of $\{u_n\}$ with $\Delta - \lim_{n \rightarrow \infty} v_n = v \in K$. Then by Theorem 4.9 we have $v \in F(T)$ and by Lemma 2.13 we conclude that $u = v$, hence we get $\omega_w(x_n) \subseteq F(T)$. Let take subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. Because of $v \in \omega_w(x_n) \subseteq F(T)$, $\{d(x_n, u)\}$ converges, so by Lemma 2.13 we have $x = u$, this means that $\omega_w(x_n)$ include exactly one point. \square

Theorem 4.15. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow C(X)$ be a generalized multivalued hybrid mapping type I with $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ for all $x \in K$ where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$, $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. If $\{x_n\}$ is a sequence in K defined by (1.1) with $\liminf_{n \rightarrow \infty} \alpha_n[(1 - \alpha_n)\frac{R}{2} - \frac{2k_2(x)}{1-a_3(x)}] > 0$ and $\liminf_{n \rightarrow \infty} \beta_n[(1 - \beta_n)\frac{R}{2} - \frac{2k_2(x)}{1-a_3(x)}] > 0$ then $\{x_n\}$ have a Δ -limit which in $F(T)$.*

Proof. Let $p \in F(T)$ then for any $x \in K, u \in Tx$ we have that

$$d^2(u, p) \leq d^2(x, p) + \frac{k_2(p)}{1 - a_3(p)} d^2(u, x)$$

since metric projection P_K is nonexpansive by Lemma 2.16, and $P_K(p) = \{x \in K : d(p, x) = d(p, K)\} = \{p\}$ we have

$$\begin{aligned}
d^2(y_n, p) &= d^2(P_K((1 - \beta_n)x_n \oplus \beta_n v_n), P_K(p)) \\
&\leq d^2((1 - \beta_n)x_n \oplus \beta_n v_n, p) \\
&\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(v_n, p) \\
&\quad - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, v_n) \\
&\leq (1 - \beta_n)d^2(x_n, p) + \beta_n [d^2(x, p) + \frac{k_2(p)}{1 - a_3(p)} d^2(v_n, x)] \\
&\quad - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, v_n) \\
&\leq (1 - \beta_n)d^2(x_n, p) + \beta_n (d^2(x_n, p) \\
&\quad + \frac{k_2(p)}{1 - a_3(p)} d^2(v_n, x_n)) - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, v_n) \\
&\leq d^2(x_n, p) + \beta_n (\frac{k_2(p)}{1 - a_3(p)} - \frac{R}{2}(1 - \beta_n)) d^2(v_n, x_n) \\
&\leq d^2(x_n, p)
\end{aligned}$$

and

$$\begin{aligned}
d^2(x_{n+1}, p) &= d^2(P_K((1 - \alpha_n)y_n \oplus \alpha_n u_n), P_K(p)) \\
&\leq d^2((1 - \alpha_n)y_n \oplus \alpha_n u_n, p) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n d^2(u_n, p) \\
&\quad - \frac{R}{2}(1 - \alpha_n)\alpha_n d^2(y_n, u_n) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n [d^2(y, p) + \frac{k_2(p)}{1 - a_3(p)}d^2(u_n, y)] \\
&\quad - \frac{R}{2}(1 - \alpha_n)\alpha_n d^2(y_n, u_n) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n (d^2(y_n, p) \\
&\quad + \frac{k_2(p)}{1 - a_3(p)}d^2(u_n, y_n)) - \frac{R}{2}(1 - \alpha_n)\alpha_n d^2(y_n, u_n) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n (d^2(y_n, p)) \\
&\quad + \alpha_n (\frac{k_2(p)}{1 - a_3(p)} - \frac{R}{2}(1 - \alpha_n))d^2(u_n, y_n) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n (d^2(y_n, p)) \\
&\quad + \alpha_n (\frac{k_2(p)}{1 - a_3(p)} - \frac{R}{2}(1 - \alpha_n))d^2(u_n, y_n) \\
&\leq d^2(y_n, p) + \alpha_n (\frac{k_2(p)}{1 - a_3(p)} - \frac{R}{2}(1 - \alpha_n))d^2(u_n, y_n) \\
&\leq d^2(y_n, p) \\
&\leq d^2(x_n, p).
\end{aligned}$$

Here we have $d^2(x_{n+1}, p) \leq d^2(x_n, p)$ implies that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists, and $d(x_{n+1}, p) \leq d(y_n, p) \leq d(x_n, p)$ implies $\lim_{n \rightarrow \infty} [d(x_n, p) - d(y_n, p)] = 0$. Since $\beta_n (\frac{k_2(p)}{1 - a_3(p)} - \frac{R}{2}(1 - \beta_n))d^2(v_n, x_n) \leq d^2(x_n, p) - d^2(y_n, p)$, by assumption we have that $\lim_{n \rightarrow \infty} d^2(v_n, x_n) = 0$, so $\lim_{n \rightarrow \infty} d(v_n, x_n) = 0$, $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$. Hence, by Lemma 4.12, $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ include exactly one point. This means that $\{x_n\}$ have a Δ -limit which in $F(T)$ \square

Theorem 4.16. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty compact convex subset of X , and $T : K \rightarrow C(X)$ be a continuous generalized multivalued hybrid mapping type I with $\frac{2k_1(x)}{1 - a_2(x)} < \frac{R}{2}$ for all $x \in K$ where $R = (\pi - 2\varepsilon) \tan \varepsilon$, $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. If $\{x_n\}$ is a sequence in K defined by (1.1) with $\liminf_{n \rightarrow \infty} \alpha_n [(1 - \alpha_n) \frac{R}{2} - \frac{2k_2(x)}{1 - a_3(x)}] > 0$ and $\liminf_{n \rightarrow \infty} \beta_n [(1 - \beta_n) \frac{R}{2} - \frac{2k_2(x)}{1 - a_3(x)}] > 0$ then $\{x_n\}$ have a Δ -limit which in $F(T)$.*

Proof. By Theorem 4.15, we have that $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in F(T)$. Since K is compact there is a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$, say $\lim_{i \rightarrow \infty} x_{n_i} = z$. Then we have

$$d(z, Tz) \leq d(z, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Tz)$$

and taking limit on i , continuity of T implies that $z \in Tz$. \square

Theorem 4.17. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow C(X)$ be a (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I with $T(p) = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence in K defined by (1.2) have a Δ -limit which in $F(T)$*

Proof. Let $p \in F(T)$ then $T(p) = \{p\}$ since T is a (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I, for all $x \in K, u \in Tx$ we have that

$$\begin{aligned} d^2(u, p) &\leq a_1(x)d^2(u, p) + a_2(x)d^2(u, p) \\ &\leq b_1(x)d^2(p, x) + b_2(x)d^2(p, x) \\ &\leq d^2(p, x) \end{aligned}$$

Hence we have that $d(u, p) \leq d(x, p)$. Then

$$\begin{aligned} d(z_n, p) &= d(P_K((1 - \beta_n)x_n \oplus \beta_n w_n), P_K(p)) \\ &= d(P_K((1 - \beta_n)x_n \oplus \beta_n w_n), p) \\ &\leq d((1 - \beta_n)x_n \oplus \beta_n w_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(w_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq d(x_n, p) \end{aligned}$$

and

$$\begin{aligned} d(y_n, p) &= d(P_K((1 - \alpha_n)w_n \oplus \alpha_n v_n), p) \\ &\leq d(P_K((1 - \alpha_n)w_n \oplus \alpha_n v_n), P_K(p)) \\ &\leq d((1 - \alpha_n)w_n \oplus \alpha_n v_n, p) \\ &\leq (1 - \alpha_n)d(w_n, p) + \alpha_n d(v_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(z_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(z_n, p) \\ &\leq d(x_n, p) \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(P_K(u_n), P_K(p)) \\ &\leq d(u_n, p) \\ &\leq d(y_n, p) \end{aligned}$$

so by $d(x_{n+1}, p) \leq d(y_n, p) \leq d(x_n, p)$ implies $\lim_{n \rightarrow \infty} d(x_n, p) = \lim_{n \rightarrow \infty} d(y_n, p)$ exist. Let say $\lim_{n \rightarrow \infty} d(x_n, p) = k$. Since $d(w_n, p) \leq d(x_n, p)$ and $d(v_n, p) \leq d(z_n, p) \leq d(x_n, p)$ we have that $\limsup_{n \rightarrow \infty} d(w_n, p) \leq k, \limsup_{n \rightarrow \infty} d(v_n, p) \leq k$ and

$$\begin{aligned}
d(y_n, p) &= d(P_K((1 - \alpha_n)w_n \oplus \alpha_n v_n, p)) \\
&\leq d(P_K((1 - \alpha_n)w_n \oplus \alpha_n v_n, P_K(p))) \\
&\leq d((1 - \alpha_n)w_n \oplus \alpha_n v_n, p) \\
&\leq (1 - \alpha_n)d(w_n, p) + \alpha_n d(v_n, p) \\
&\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(z_n, p) \\
&\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(z_n, p) \\
&\leq d(x_n, p)
\end{aligned}$$

implies that $\lim_{n \rightarrow \infty} d((1 - \alpha_n)w_n + \alpha_n v_n, p) = k$, so by Lemma 2.3 we have that $\lim_{n \rightarrow \infty} d(w_n, v_n) = 0$. And again from

$$\begin{aligned}
d(y_n, p) &= d(P_K((1 - \alpha_n)w_n \oplus \alpha_n v_n, p)) \\
&\leq d(P_K((1 - \alpha_n)w_n \oplus \alpha_n v_n, P_K(p))) \\
&\leq d((1 - \alpha_n)w_n \oplus \alpha_n v_n, p) \\
&\leq (1 - \alpha_n)d(w_n, p) + \alpha_n d(v_n, p) \\
&\leq (1 - \alpha_n)(d(w_n, v_n) + d(v_n, p)) + \alpha_n d(v_n, p) \\
&\leq (1 - \alpha_n)d(w_n, v_n) + d(v_n, p)
\end{aligned}$$

we have that $k \leq \liminf_{n \rightarrow \infty} d(v_n, p)$, and since $d(v_n, p) \leq d(z_n, p) \leq d(x_n, p)$ we have that $\lim_{n \rightarrow \infty} d(x_n, p) = k$. By CN^* inequality we have

$$\begin{aligned}
d^2(z_n, p) &= d^2(P_K((1 - \beta_n)x_n \oplus \beta_n w_n), P_K(p)) \\
&= d^2(P_K((1 - \beta_n)x_n \oplus \beta_n w_n), p) \\
&\leq d^2((1 - \beta_n)x_n \oplus \beta_n w_n, p) \\
&\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(w_n, p) - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, w_n) \\
&\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(x_n, p) - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, w_n) \\
&\leq d^2(x_n, p) - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, w_n)
\end{aligned}$$

implies that

$$\frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, w_n) \leq d^2(x_n, p) - d^2(z_n, p).$$

Since $\lim_{n \rightarrow \infty} (d^2(x_n, p) - d^2(z_n, p)) = 0$ and $\inf_n (1 - \beta_n)\beta_n > 0$ then we have that $\lim_{n \rightarrow \infty} d(x_n, w_n) = 0$ and so. $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then by Lemma 4.13 $\{x_n\}$ have Δ -limit which in $F(T)$. \square

Theorem 4.18. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty compact convex subset of X , and $T : K \rightarrow KC(X)$ be a continuous (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type I with $T(p) = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence in K defined by (1.2) have a Δ -limit which in $F(T)$*

Proof. Proof is similar to Theorem 4.16. \square

Theorem 4.19. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow KC(X)$ be a generalized multivalued hybrid mapping type II with $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ for all $x \in K$ where $R = (\pi - 2\varepsilon) \tan \varepsilon$, $F(T) \neq \emptyset$ and $TP = \{p\}$ for all $p \in F(T)$. If $\{x_n\}$ is a sequence in K defined by (1.1) with $\liminf_{n \rightarrow \infty} \alpha_n[(1 - \alpha_n)\frac{R}{2} - \frac{2k_2(x)}{1-a_3(x)}] > 0$ and $\liminf_{n \rightarrow \infty} \beta_n[(1 - \beta_n)\frac{R}{2} - \frac{2k_2(x)}{1-a_3(x)}] > 0$ then $\{x_n\}$ have a Δ -limit which in $F(T)$*

Proof. Let $p \in F(T)$ then for any $x \in K$, we have that

$$H^2(Tx, Tp) \leq d^2(x, p) + \frac{k_2(p)}{1 - a_3(p)} d^2(Tx, x)$$

since metric projection P_K is nonexpansive by Lemma 2.16, and $P_K(p) = \{x \in K : d(p, x) = d(p, K)\} = \{p\}$ we have

$$\begin{aligned} d^2(y_n, p) &= d^2(P_K((1 - \beta_n)x_n \oplus \beta_n v_n), P_K(p)) \\ &\leq d^2((1 - \beta_n)x_n \oplus \beta_n v_n, p) \\ &\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(v_n, p) \\ &\quad - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, v_n) \\ &\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(v_n, Tp) \\ &\quad - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, Tx_n) \\ &\leq (1 - \beta_n)d^2(x_n, p) + \beta_n H^2(Tx_n, Tp) \\ &\quad - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, Tx_n) \\ &\leq (1 - \beta_n)d^2(x_n, p) + \beta_n (d^2(x_n, p) \\ &\quad + \frac{k_2(p)}{1 - a_3(p)} d^2(Tx_n, x_n)) - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, Tx_n) \\ &\leq d^2(x_n, p) + \beta_n (\frac{k_2(p)}{1 - a_3(p)} - \frac{R}{2}(1 - \beta_n)) d^2(Tx_n, x_n) \\ &\leq d^2(x_n, p) \end{aligned}$$

and

$$\begin{aligned}
d^2(x_{n+1}, p) &= d^2(P_K((1 - \alpha_n)y_n \oplus \alpha_n u_n), P_K(p)) \\
&\leq d^2((1 - \alpha_n)y_n \oplus \alpha_n u_n, p) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n d^2(u_n, p) \\
&\quad - \frac{R}{2}(1 - \alpha_n)\alpha_n d^2(y_n, u_n) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n d^2(u_n, Tp) \\
&\quad - \frac{R}{2}(1 - \alpha_n)\alpha_n d^2(y_n, Ty_n) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n H^2(Ty_n, Tp) \\
&\quad - \frac{R}{2}(1 - \alpha_n)\alpha_n d^2(y_n, Ty_n) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n(d^2(y_n, p) \\
&\quad + \frac{k_2(p)}{1 - a_3(p)}d^2(Ty_n, y_n)) - \frac{R}{2}(1 - \alpha_n)\alpha_n d^2(y_n, Ty_n) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n(d^2(y_n, p) \\
&\quad + \alpha_n(\frac{k_2(p)}{1 - a_3(p)} - \frac{R}{2}(1 - \alpha_n))d^2(Ty_n, y_n)) \\
&\leq (1 - \alpha_n)d^2(y_n, p) + \alpha_n(d^2(y_n, p) \\
&\quad + \alpha_n(\frac{k_2(p)}{1 - a_3(p)} - \frac{R}{2}(1 - \alpha_n))d^2(Ty_n, y_n)) \\
&\leq d^2(y_n, p) + \alpha_n(\frac{k_2(p)}{1 - a_3(p)} - \frac{R}{2}(1 - \alpha_n))d^2(Ty_n, y_n) \\
&\leq d^2(y_n, p) \\
&\leq d^2(x_n, p).
\end{aligned}$$

Here we have $d^2(x_{n+1}, p) \leq d^2(x_n, p)$ implies that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists, and $d(x_{n+1}, p) \leq d(y_n, p) \leq d(x_n, p)$ implies $\lim_{n \rightarrow \infty} [d(x_n, p) - d(y_n, p)] = 0$. Since $\beta_n(\frac{k_2(p)}{1 - a_3(p)} - \frac{R}{2}(1 - \beta_n))d^2(Tx_n, x_n) \leq d^2(x_n, p) - d^2(y_n, p)$, by assumption we have that $\lim_{n \rightarrow \infty} d^2(Tx_n, x_n) = 0$, so $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$. Hence, by Lemma 4.14, $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ include exactly one point. This means that $\{x_n\}$ have a Δ -limit which in $F(T)$ \square

Theorem 4.20. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty compact convex subset of X , and $T : K \rightarrow KC(X)$ be a generalized multivalued hybrid mapping type II with $\frac{2k_1(x)}{1 - a_2(x)} < \frac{R}{2}$ for all $x \in K$ where $R = (\pi - 2\varepsilon)\tan \varepsilon$, $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. If $\{x_n\}$ is a sequence in K defined by (1.1) with $\liminf_{n \rightarrow \infty} \alpha_n[(1 - \alpha_n)\frac{R}{2} - \frac{2k_2(x)}{1 - a_3(x)}] > 0$ and*

$\liminf_{n \rightarrow \infty} \beta_n[(1 - \beta_n)\frac{R}{2} - \frac{2k_2(x)}{1 - a_3(x)}] > 0$ then $\{x_n\}$ have a Δ -limit which in $F(T)$

Proof. Proof is similar to Theorem 4.16. \square

Theorem 4.21. Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $\text{diam}(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X , and $T : K \rightarrow KC(X)$ be a (a_1, a_2, b_1, b_2) -multivalued hybrid mapping type II with $F(T) \neq \emptyset$, $Tp = \{p\}$ for all $p \in F(T)$ and $a_1(x) \geq 1$ for all $x \in C$. Let $\{x_n\}$ be a sequence in K defined by (1.2) strongly converges to a point of $F(T)$

Proof. Let $p \in F(T)$ then $T(p) = \{p\}$ since T is a (a_1, a_2, b_1, b_2) -multivalued hybrid mapping we have that

$$\begin{aligned} d^2(Tx, p) &\leq a_1(x)d^2(Tx, p) + a_2(p)d^2(Tx, p) \\ &\leq a_1(x)H^2(Tx, Tp) + a_2(x)d^2(Tx, p) \\ &\leq b_1(x)d^2(x, p) + b_2(x)d^2(x, p) \\ &\leq b_1(x)d^2(x, p) + b_2(x)d^2(x, p) \\ &\leq d^2(x, p) \end{aligned}$$

Assume that $a_2(x) \geq 0$ for all $x \in C$ since T is a (a_1, a_2, b_1, b_2) -multivalued hybrid mapping we have that

$$\begin{aligned} a_1(x)H^2(Tx, Tp) &\leq b_1(x)d^2(x, p) + b_2(x)d^2(x, p) - a_2(x)d^2(Tx, p) \\ &\leq d^2(x, p) \end{aligned}$$

so we have

$$H^2(Tx, Tp) \leq \frac{1}{a_1(x)}d^2(x, p) \leq d^2(x, p)$$

And now if $a_2(x) \leq 0$ for all $x \in C$ then since $a_1(x) + a_2(x) \geq 1$, $1 \geq \frac{1}{a_1(x)} - \frac{a_2(x)}{a_1(x)}$ and since T is a (a_1, a_2, b_1, b_2) -multivalued hybrid mapping we have that

$$\begin{aligned} a_1(x)H^2(Tx, Tp) &\leq b_1(x)d^2(x, p) + b_2(x)d^2(x, p) - a_2(x)d^2(Tx, p) \\ &\leq b_1(x)d^2(x, p) + b_2(x)d^2(x, p) - a_2(x)d^2(Tx, p) \\ &\leq d^2(x, p) - a_2(x)d^2(Tx, p) \end{aligned}$$

which implies that

$$\begin{aligned} H^2(Tx, Tp) &\leq \frac{1}{a_1(x)}d^2(x, p) - \frac{a_2(p)}{a_1(p)}d^2(Tx, p) \\ &\leq \frac{1}{a_1(x)}d^2(x, p) - \frac{a_2(x)}{a_1(x)}d^2(x, p) \\ &= \left(\frac{1}{a_1(x)} - \frac{a_2(x)}{a_1(x)}\right)d^2(p, x) \\ &\leq d^2(p, x). \end{aligned}$$

Hence we have that $H(Tp, Tx) \leq d(p, x)$.

$$\begin{aligned}
d(z_n, p) &= d(P_K((1 - \beta_n)x_n \oplus \beta_n w_n), P_K(p)) \\
&= d(P_K((1 - \beta_n)x_n \oplus \beta_n w_n), p) \\
&\leq d((1 - \beta_n)x_n \oplus \beta_n w_n, p) \\
&\leq (1 - \beta_n)d(x_n, p) + \beta_n d(w_n, p) \\
&\leq (1 - \beta_n)d(x_n, p) + \beta_n d(w_n, Tp) \\
&\leq (1 - \beta_n)d(x_n, p) + \beta_n H(Tx_n, Tp) \\
&\leq (1 - \beta_n)d(x_n, p) + \beta_n d(x_n, p) \\
&\leq d(x_n, p)
\end{aligned}$$

and

$$\begin{aligned}
d(y_n, p) &= d(P_K((1 - \alpha_n)w_n \oplus \alpha_n v_n), p) \\
&\leq d(P_K((1 - \alpha_n)w_n \oplus \alpha_n v_n), P_K(p)) \\
&\leq d((1 - \alpha_n)w_n \oplus \alpha_n v_n, p) \\
&\leq d((1 - \alpha_n)w_n \oplus \alpha_n v_n), p) \\
&\leq (1 - \alpha_n)d(w_n, p) + \alpha_n d(v_n, p) \\
&\leq (1 - \alpha_n)d(w_n, Tp) + \alpha_n d(v_n, Tp) \\
&\leq (1 - \alpha_n)H(Tx_n, p) + \alpha_n H(Tz_n, Tp) \\
&\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(z_n, p) \\
&\leq d^2(z_n, p)
\end{aligned}$$

and

$$\begin{aligned}
d^2(x_{n+1}, p) &= d(P_K(u_n), P_K(p)) \\
&\leq d(u_n, p) \\
&\leq H(Ty_n, Tp) \\
&\leq d(y_n, p)
\end{aligned}$$

so by $d(x_{n+1}, p) \leq d(y_n, p) \leq d(z_n, p) \leq d(x_n, p)$ implies $\lim_{n \rightarrow \infty} d(x_n, p)$ exist. By CN* inequality we have

$$\begin{aligned}
d^2(z_n, p) &= d^2(P_K((1 - \beta_n)x_n \oplus \beta_n w_n), P_K(p)) \\
&= d^2(P_K((1 - \beta_n)x_n \oplus \beta_n w_n), p) \\
&\leq d^2((1 - \beta_n)x_n \oplus \beta_n w_n, p) \\
&\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(w_n, p) - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, w_n) \\
&\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(w_n, Tp) - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, Tx_n) \\
&\leq (1 - \beta_n)d^2(x_n, p) + \beta_n H^2(Tx_n, Tp) - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, Tx_n) \\
&\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(x_n, p) - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, Tx_n) \\
&\leq d^2(x_n, p) - \frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, Tx_n)
\end{aligned}$$

implies that

$$\frac{R}{2}(1 - \beta_n)\beta_n d^2(x_n, Tx_n) \leq d^2(x_n, p) - d^2(z_n, p).$$

Since $\lim_{n \rightarrow \infty} (d(x_n, p) - d^2(z_n, p)) = 0$ and $\inf_n (1 - \beta_n)\beta_n > 0$ then we have that $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Rest of the proof is similar to Theorem 4.16.

Then by Lemma 4.5 $\{x_n\}$ have Δ -limit which in $F(T)$. The above discussion will hold if $a_2(p) \geq 0$ \square

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